Abstract. The purpose of this paper is to understand in more detail the shape of the eigenvectors of the random Schrödinger operator $H = \Delta + V$ on $\ell^2(\mathbb{Z})$. Here Δ is the discrete Laplacian and V is a random potential. It is well known that under certain assumptions on V the spectrum of this operator is pure point and its eigenvectors are exponentially localized; a phenomenon known as Anderson Localization. We restrict the operator to $\mathbb{Z}_n$ and consider the critical model,

$$(H_n\psi)_\ell = \psi_{\ell-1,n} + \psi_{\ell+1,n} + v_{\ell,n}\psi_\ell, \quad \psi_0 = \psi_{n+1} = 0,$$

with $v_k = \sigma\omega_k/\sqrt{n}$ and ω_k independent random variables with mean 0 and variance 1. We characterize the scaling limit of the shape of a uniformly chosen eigenvector of H_n. We show that it converges in law to

$$\exp\left(-\frac{|t-u|}{4} + \frac{B_{|t-u|}}{\sqrt{2}}\right)$$

on $[0,1]$. Here u is uniform on $[0,1]$ and B_t is an independent two sided Brownian motion started form 0.

1. Introduction

We consider the critical model of one-dimensional discrete random Schrödinger operators given by the matrix

$$H_n = \begin{pmatrix} v_{1,n} & 1 & & & & \\ 1 & v_{2,n} & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & v_{n-1,n} & 1 \\ & & & & 1 & v_{n,n} \end{pmatrix} \tag{1.1}$$ {shrod1dmatrix}

where

$$v_{k,n} = \sigma\omega_k/\sqrt{n}. \tag{1.2}$$

Here ω_k are independent random variables with mean 0, variance 1 and bounded third absolute moment.

If there is no noise (i.e. $\sigma = 0$) then the eigenvalues μ_k and eigenvectors ψ_k of H_n are given by

$$\mu_k = 2\cos(\pi k/(n+1)),$$
$$\psi_k(\ell) = \sin(\pi k\ell/(n+1)).$$

The asymptotic density near $E \in (-2,2)$ is given by the arcsin law, $\frac{\rho}{2\pi}$ with

$$\rho = \rho(E) = \frac{1}{\sqrt{1-E^2/4}}\mathbf{1}_{|E|<2}. \tag{1.3}$$ {defrho}

The fact that the eigenvectors of H_n are delocalized was shown in [KVV12]. Building on the framework developed in that paper, here we focus on actual scaling limits of the eigenvectors of H_n. The eigenvectors are highly oscillatory and so we focus on their induced L^2 measure. For μ an eigenvalue of H_n and ψ^μ the corresponding normalized eigenvector ($\sum_{\ell=1}^{n} |\psi^\mu(\ell)|^2 = 1$), we consider the measure on $[0,1]$ whose density is

$$|\psi^\mu(\lfloor nt \rfloor)|^2 \, dt.$$

We let $\mathcal{M}([0,1])$ be the space of finite measures on $[0,1]$ with the weak topology. By this we mean that $\mu_n \to \mu$ if $\int f d\mu_n \to \int f d\mu$ for every $f \in C_b([0,1], \mathbb{R})$.
Our main result is a statement about the joint convergence in law of the pairs

$$\left(\mu, |\psi^\mu(\lfloor nt \rfloor)|^2 \, dt\right) \in \mathbb{R} \times \mathcal{M}[0,1]$$

when we pick μ uniformly at random from the eigenvalues of H_n.

{global_evector}

Theorem 1.1. *Let $\mathcal{B}$ be a standard two-sided Brownian motion started from 0 and take*

$$S(t) = \exp\left(\frac{\mathcal{B}_t}{\sqrt{2}} - \frac{|t|}{4}\right).$$

Pick μ uniformly from the eigenvalues of H_n and let ψ^μ be the corresponding normalized eigenvector. Then letting $\tau(E) = (\sigma\rho(E))^2$,

$$\left(\mu, n\,|\psi^\mu(\lfloor nt \rfloor)|^2 \, dt\right) \Rightarrow \left(E, \frac{S\big(\tau(t-u)\big)dt}{\int_0^1 ds\, S\left(\tau(s-u)\right)}\right),$$

where E is distributed according to the arcsin law and u is an independent uniform from $[0,1]$.

The proof relies on the scaling limit of the transfer matrix framework for this problem that was developed in [KVV12]. The organization of this paper is the following. In section 2 we explain the transfer matrix framework along with the main theorem of [KVV12] along with our slight modification. In Section 3, we give a local version of Theorem 1.1. And finally in Section 4 we show how this local result gives the proof of the main theorem.

{TransferMatrix}

2. Transfer Matrix

[KVV12] showed that the transfer matrix framework has a limiting evolution; it is this limiting object that enabled them to characterize the limiting eigenvalue process. Our main technical result is a slight strengthening of the convergence in that theorem. Our analysis will make use of this convergence and the correspondence between eigenvectors and transfer matrices. In order to state that theorem we first introduce the transfer matrix description of the spectral problem for H_n.

We can write the eigenvalue equation $H_n\psi = \mu\psi$ or

$$\psi(\ell-1) + v_{\ell,n}\psi(\ell) + \psi(\ell+1) = \mu\psi(\ell),$$

as the recursion $\psi(\ell+1) = (\mu - v_{\ell,n})\psi(\ell) - \psi(\ell-1)$ with μ an eigenvalue of H_n when $\psi(0) = 0 = \psi(n+1)$. We write this as,

{txmat_form}

$$\begin{pmatrix} \psi(\ell+1) \\ \psi(\ell) \end{pmatrix} = T(\mu - v_{\ell,n}) \begin{pmatrix} \psi(\ell) \\ \psi(\ell-1) \end{pmatrix} = M_n^\mu(\ell) \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}, \tag{2.1}$$

where

$$T(x) := \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \text{ and } M_n(\mu, \ell) := T(\mu - v_{\ell,n})T(\mu - v_{\ell-1,n}) \cdots T(\mu - v_{1,n}).$$

Then μ is an eigenvalue of H_n if and only if

$$M_n(\mu, n) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.2}$$

{ev_cond}

for some $c \in \mathbb{R}$ or, equivalently $(M_n(\mu, n))_{11} = 0$. Moreover, notice that the corresponding normalized eigenvector ψ^μ is given by

$$\psi^\mu(\ell) = \frac{m_n^\mu(\ell - 1)}{\sqrt{\sum_{k=1}^n |m_n^\mu(k)|^2}}, \quad \ell = 1, \dots, n, \tag{2.3}$$

{evector_condn}

where we have written $m_n^\mu(\ell) = (M_n(\mu, \ell))_{11}$.

For local analysis in view of (1.3) we parametrize $\mu = E + \frac{\lambda}{\rho(E)n}$. We will use the notation $M_{n,E}(\lambda, \ell)$ to emphasize dependence on λ and E, and use the similar notation for other quantities. Sometimes we will drop E from our notation and when we do so we are implicity assuming that there is a fixed $E \in (-2, 2)$ in the background. Setting

$$\epsilon_{\ell,n} = \frac{\lambda}{\rho n} - \frac{\sigma \omega_\ell}{\sqrt{n}}, \tag{2.4}$$

{epsn}

we have

$$M_{n,E}(\lambda, \ell) = T(E + \epsilon_{\ell,n})T(E + \epsilon_{\ell-1,n}) \cdots T(E + \epsilon_{1,n}) \text{ for } 0 \le \ell \le n. \tag{2.5}$$

{Mn}

As $T(E+\epsilon_{\ell,n})$ is a perturbation of $T(E)$, we follow the evolution in the coordinates that diagonalize $T(E)$. For $|E| < 2$, we can write $T(E) = ZDZ^{-1}$ with

$$D = \begin{pmatrix} \overline{z} & 0 \\ 0 & z \end{pmatrix}, \quad Z = \frac{i\rho(E)}{2} \begin{pmatrix} \overline{z} & z \\ 1 & 1 \end{pmatrix}, \quad z = E/2 + i\sqrt{1 - (E/2)^2}. \tag{2.6}$$

{zdef}

From this we can see that for $|E| < 2$, $M_{n,E}(\lambda, \ell)$ is a perturbation of the rotation matrix D^ℓ and so we cannot hope for a limiting process. However, if we regularize the evolution by undoing the rotation and consider instead

$$Q_{n,E}(\lambda, \ell) = T^{-\ell}(E) M_{n,E}(\lambda, \ell), \tag{2.7}$$

{reg_transfer}

then we have the the following scaling limit from [KVV12].

{DiffusionTransfe

Theorem 2.1. *Assume $0 < |E| < 2$. Let $\mathcal{B}(t), \mathcal{B}_2(t), \mathcal{B}_3(t)$ be independent standard Brownian motions in $\mathbb{R}$, $\mathcal{W}(t) = \frac{1}{\sqrt{2}}(\mathcal{B}_2(t) + i\mathcal{B}_3(t))$. Then the stochastic differential equation*

$$dQ(\lambda, t) = \frac{1}{2} Z \left(\begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix} dt + \begin{pmatrix} id\mathcal{B} & d\mathcal{W} \\ d\overline{\mathcal{W}} & -id\mathcal{B} \end{pmatrix} \right) Z^{-1} Q(\lambda, t), \qquad Q(\lambda, 0) = I \tag{2.8}$$

{LimitingTransfer

has a unique strong solution $Q(\lambda, t) : \lambda \in \mathbb{C}, t \ge 0$, which is analytic in λ.

Moreover, let $\tau = (\sigma\rho(E))^2$, *then*

$$\left(Q_{n,E}\left(\lambda, \lfloor nt/\tau \rfloor\right), 0 \le t \le \tau\right) \Rightarrow (Q(\lambda/\tau, t), 0 \le t \le \tau),$$

in the sense of finite dimensional distributions for λ *and uniformly in* t. *Moreover, the random analytic functions* $Q_{n,E}(\lambda, t)$ *converge in law to* $Q(\lambda/\tau, t)$ *with respect to the local uniform topology on* $\mathbb{C} \times [0, \tau]$.

Remark 2.1. *The main part of this theorem is proven in [KVV12]. The work we have done here is to strengthen the tightness argument which allows us to get convergence in law with respect to the local uniform topology on* $\mathbb{C} \times [0, \tau]$. *The extra tightness argument along with how this implies the result is in Section 5.*

LocalConvergence}

3. Local Limits of Eigenvalue-Eigenvector Pairs

In this section we prove a local version of Theorem 1.1. We will zoom in on the eigenvalue point process around a fixed $0 < |E| < 2$. From Equation (1.3) we see that the eigenvalue spacings near E are like $1/(n\rho(E))$ and so we consider the operator $n\rho(E)(H_n - E)$ and its eigenvalues $\Lambda_{n,E}$. Our local result is about the joint convergence of eigenvalue, eigenvectors pairs of this scaled operator. As with our global limit we consider the induced L^2 measure on $[0, \tau]$ coming from the eigenvector since it is otherwise too irregular to have a scaling limit. We think of these pairs as a point process on $X = \mathbb{R} \times \mathcal{M}[0, \tau]$,

$$\mathcal{P}_{n,E} = \left\{ \left(n\rho(E)(\mu - E) + \theta, \frac{n}{\tau} \left|\psi^\mu(\lfloor nt/\tau \rfloor)\right|^2 dt \right) : \mu \text{ an eigenvalue of } H_n \right\}.$$

With the usual product topology X is a complete, separable metric space. Let $\mathcal{M}(X)$ be the set of locally finite measures on X with the local weak topology. In other words, we say $\mu_n \in \mathcal{M}(X)$ converges to $\mu \in \mathcal{M}(X)$ if for every continuous function $f : X \to \mathbb{R}$ with compact support, $\int f d\mu_n \to \int \psi d\mu$. A random measure on $\mathcal{M}(X)$ is a measurable map $\omega \to \mu \in \mathcal{M}(X)$, with the Borel σ-algebra on $\mathcal{M}(X)$. By the point process $\mathcal{P}_{n,E}$ we mean the random measure in $\mathcal{M}(X)$ given by the sum of the delta masses corresponding to points in the set. And by convergence in law of a sequence of point processes on X we mean the usual notion of weak convergence of the corresponding random measures on $\mathcal{M}(X)$.

{local_evector}

Theorem 3.1. *Fix* $0 < |E| < 2$ *and take* $\tau = \tau(E) = (\sigma\rho(E))^2$. *Let* θ *be uniform on* $[0, 2\pi]$. *Then, the point process on* $\mathbb{R} \times \mathcal{M}[0, \tau]$

$$\left\{ \left(n\rho(E)(\mu - E) + \theta, \frac{n}{\tau} \left|\psi^\mu(\lfloor nt/\tau \rfloor)\right|^2 dt \right) : \mu \textit{ an eigenvalue of } H_n \right\}$$

converges in law to a point process $\mathcal{P}_E$.

Moreover, for $t \in \mathbb{R}$, *let*

$$S(t) = \exp\left(\mathcal{Z}_t/\sqrt{2} - |t|/4\right),$$

where $\mathcal{Z}$ is a two sided Brownian motion started from 0.
And define a measure μ_E on X such that for every $F \in C_b(\mathbb{R} \times \mathcal{M}[0,\tau])$,

$$\int F(\lambda,\nu)\, d\mu_E(\lambda,\nu) = \frac{1}{2\pi}\int d\lambda \mathbf{E} F\left(\lambda, \frac{S(t-u)dt}{\int_0^1 ds\, S(s-u)}\right),$$

with u independent, uniform on $[0,\tau]$. Then the intensity measure of $\mathcal{P}_E$ is μ_E.

Remark 3.2. *We note that [KVV12] proved the convergence of the local eigenvalue point process and characterized the limit. Our result is an extension to the eigenvalue-eigenvector pairs.*

The proof of weak convergence proceeds in the usual steps. We first show subsequential convergence and then that the limit does not depend on the subsequence. We calculate the intensity measure in a separate lemma.

In order to characterize the limiting point process, we introduce two limiting random processes. Note that for $0 < |E| < 2$, for any $a, b \in \mathbb{R}^2$ we have

$$Z^{-1}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} i(a-bz) \\ \overline{i(a-bz)}. \end{pmatrix},$$

So Z^{-1} maps real vectors to vectors with conjugate entries. Since for $\lambda \in \mathbb{R}$ the transfer matrix $Q_{n,E}(\lambda,\ell)$ is real valued the process $Q(\lambda,t)$ will also be real valued. Therefore, we can write for $\lambda \in \mathbb{R}$,

$$\begin{pmatrix} iq^\lambda(t) \\ \overline{iq^\lambda(t)} \end{pmatrix} := Z^{-1}Q(\lambda,t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.1}$$ {Qtoq}

for some complex numbers $q^\lambda(t)$ where $q^\lambda(0) = 1$ (the extra i in the above definition makes this and some upcoming formulas nicer). We will show that q^λ determines both the limiting eigenvalue point process and the limiting eigenvector shape. It will be useful to write $q = re^{i\theta}$ in its polar coordinates and so we make the following definition/lemma.

{SDErtheta}

Lemma 3.3. *For $\lambda \in \mathbb{R}$, we define $\theta^\lambda(t) := 2\arg q^\lambda(t)$ and $r^\lambda(t) := \ln\left|q^\lambda(t)\right|^2$. Then r and θ are well defined and uniquely satisfy the following stochastic differential equations,*

$$d\theta^\lambda(t) = \lambda dt + d\mathcal{B} + \operatorname{Im}\left[e^{-i\theta^\lambda(t)}d\mathcal{W}\right], \quad \theta^\lambda(0) = 0 \tag{3.2}$$

$$dr^\lambda(t) = \frac{dt}{4} + \operatorname{Re}\left[e^{-i\theta^\lambda(t)}d\mathcal{W}\right], \quad r^\lambda(0) = 0. \tag{3.3}$$

coupled together for all values of $\lambda \in \mathbb{R}$ where $\mathcal{B}$ and $\mathcal{W}$ are standard real and complex Brownian motions.
Moreover $\theta^\lambda(t)$ is almost surely real analytic in λ and $\phi^\lambda(t) := \frac{\partial\theta^\lambda(t)}{\partial\lambda}$ satisfies the SDE

$$d\phi^\lambda(t) = dt - \operatorname{Re}(e^{-i\phi^\lambda(t)}d\mathcal{W})\phi^\lambda(t).$$

Our first step in proving Theorem 3.1 is to show convergence in law along subsequences.

{subseq_convergenc

Lemma 3.4. *Fix $0 < |E| < 2$. For $\lambda \in \mathbb{R}$, let $\boldsymbol{m}_n^\lambda$, $\boldsymbol{q}^\lambda$ be measures on $[0, \tau]$ with densities*

$$d\boldsymbol{m}_n^\lambda(t) = \left| \Big((2/\rho(E)) M_{n,E}\left(\lambda, \lfloor nt/\tau \rfloor\right) \Big)_{11} \right|^2 dt,$$
$$d\boldsymbol{q}^\lambda(t) = \left| q^\lambda(t) \right|^2 dt.$$

Suppose that n_j is a subsequence along which $z(E)^{n_j} \to \tilde{z}$. Then, in law,

$$\left\{ \left(\lambda, \boldsymbol{m}_n^\lambda\right) : \lambda \in \Lambda_{n_j,E} \right\} \Rightarrow \left\{ \left(\lambda, 2\boldsymbol{q}^{\lambda/\tau}\right) : \lambda \in \mathsf{Sch}_\tau^{\tilde{\phi}} \right\}$$

where,

$$\mathsf{Sch}_\tau^{\tilde{\phi}} = \left\{ \lambda \in \mathbb{R} : \theta(\lambda/\tau, \tau) \in 2\pi\mathbb{Z} + 2\tilde{\phi} \right\} \text{ and } \tilde{\phi} = \arg(z - \tilde{z}).$$

The next lemma shows that the distribution of the limit does not depend on the subsequence.

{subseq_dist}

Lemma 3.5. *Fix $\tau > 0$ and u uniform in $[0, 2\pi]$. Then for any $\phi \in \mathbb{R}$,*

$$\left\{ \left(\lambda + u, \boldsymbol{q}^{\lambda/\tau}\right) : \lambda \in \mathsf{Sch}_\tau^{\phi} \right\} =^d \left\{ \left(\lambda, \boldsymbol{q}^{\lambda/\tau}\right) : \lambda \in \mathsf{Sch}_\tau^{*} \right\}.$$

And finally we need the following lemma to help calculate the intensity measure of the limiting point process.

ntensity_measure}

Lemma 3.6. *For every $G \in C_b\left(\mathbb{R} \times C[0, \tau]\right)$,*

$$\mathbf{E} \sum_{\lambda \in \mathsf{Sch}_\tau^*} G(\lambda, q^{\lambda/\tau}) = \frac{1}{2\pi} \int d\lambda\, \mathbf{E}\left[G\left(\lambda, \exp\left(\frac{\mathcal{B}}{\sqrt{2}} + \frac{f^u}{2} \right)\right)\right],$$

with $\mathcal{B}$ a standard Brownian motion started at zero, u independent, uniform on $[0, \tau]$, and $f^u(t) = \frac{1}{2}(u - |u - t|)$.

The above three lemmas give the proof of Theorem 3.1.

Proof of Theorem 3.1. Lemma 3.4 gives that along a subsequence n_j such that z^{n_j} converges to $\tilde{z}$, we have that

$$\begin{aligned} \left\{ \left(\lambda + u, \boldsymbol{m}_n^\lambda\right) : \lambda \in \Lambda_{n_j,E} \right\} &\Rightarrow \left\{ \left(\lambda + u, 2\boldsymbol{q}^{\lambda/\tau}\right) : \lambda \in \mathsf{Sch}_\tau^{\tilde{\phi}} \right\} \} \\ &=^d \left\{ \left(\lambda, 2\boldsymbol{q}^{\lambda/\tau}\right) : \lambda \in \mathsf{Sch}_\tau^{*} \right\} \end{aligned}$$

with the equality following by Lemma 3.5. Since from any subsequence we can extract a further subsequence n_j such that z^{n_j} converges, this gives that

$$\left\{ \left(\lambda + u, \boldsymbol{m}_n^\lambda\right) : \lambda \in \Lambda_{n_j,E} \right\} \Rightarrow \left\{ \left(\lambda, 2\boldsymbol{q}^{\lambda/\tau}\right) : \lambda \in \mathsf{Sch}_\tau^{*} \right\}.$$

Now recall that for $\lambda \in \Lambda_{n,E}$, $\lambda = n\rho(E)(\mu - E)$ for μ an eigenvalue of H_n and the corresponding normalized eigenvector is

$$\psi^\mu(\ell) = \frac{\left(M_{n,E}(\lambda, \ell)\right)_{11}}{\sqrt{\sum_{k=1}^n \left| \left(M_{n,E}(\lambda, k)\right)_{11} \right|^2}}, \quad \ell = 1, \ldots, n.$$

And so since $d\boldsymbol{m}_n^\lambda(t) = \left|\Big((\rho/2)M_{n,E}(\lambda,(\lfloor nt/\tau\rfloor)\Big)_{11}\right|^2 dt$,

$$\frac{n}{\tau}\left|\psi^\mu(\lfloor nt/\tau\rfloor)\right|^2 dt = \frac{d\boldsymbol{m}_n^\lambda(t)}{\boldsymbol{m}_n^\lambda[0,\tau]}.$$

Since the function from $\mathcal{M}[0,1]$ to itself given by $\mu \mapsto \mu/\mu[0,1]$ is continuous except at zero and the probability that $\boldsymbol{m}_n^\lambda \equiv 0$ is zero, this gives the convergence in law,

$$\left\{\left(n\rho(E)(\mu-E)+\theta, \frac{n}{\tau}\left|\psi^\mu(\lfloor nt/\tau\rfloor)\right|^2 dt\right) : \mu \text{ an eigenvalue of } H_n\right\} \Rightarrow \left\{\left(\lambda, \frac{\boldsymbol{q}^{\lambda/\tau}}{\boldsymbol{q}^{\lambda/\tau}([0,\tau])}\right) : \lambda \in \mathsf{Sch}_\tau^{\tilde{\phi}}\right\}$$

Now note that,

$$\frac{\exp(\mathcal{B}_t + \frac{1}{2}(u-|u-t|))}{\int_0^\tau ds \exp\left(\mathcal{B}_t + \frac{1}{2}(u-|u-t|)\right)} =^d \frac{\exp\left(\mathcal{Z}_{t-u} - |u-t|/2\right)}{\int \exp\left(\mathcal{Z}_{s-u} - |u-s|/2\right)},$$

as processes on $[0,\tau]$, where $\mathcal{B}$ is a standard Brownian motion while $\mathcal{Z}$ is a two sided Brownian motion started from zero. And so from Lemma 3.6, we have the intensity measure of the limiting point process. □

We now present the proofs of the three lemmas of this section.

{local_evector_pro

Proof of Lemma 3.4. We are trying to show convergence in law of random point measures on $X = \mathbb{R}\times\mathcal{M}[0,\tau]$. In other words, we want to show that $\mu_{n_j} = \sum_{\lambda\in\Lambda_{n_j,E}}\delta(\lambda)\delta(\boldsymbol{m}_{n_j}^\lambda)$ converges in law to $\mu = \sum_{\lambda\in\mathsf{Sch}_\tau^{\tilde{\phi}}}\delta(\lambda)\delta(\boldsymbol{q}^{\lambda/\tau})$ with respect to the local weak topology. By the general theory of point processes (see Proposition 11.1.VIII, [DVJ03]) it suffices to show that for any $h \in C_c(X,\mathbb{R})$, the real valued random variables $\int h d\mu_{n_j}$ converge in law to $\int h d\mu$.

First, for all $w \in \mathbb{C}$, we let

$$F_n(w,t) := \begin{pmatrix} F_n^1(w,t) \\ F_n^2(w,t)\end{pmatrix} := Z^{-1}Q_{n,E}(w,\lfloor nt/\tau\rfloor)\begin{pmatrix}1\\0\end{pmatrix},$$

$$F(w,t) := \begin{pmatrix} F^1(w,t) \\ F^2(w,t)\end{pmatrix} := Z^{-1}Q(w,t)\begin{pmatrix}1\\0\end{pmatrix}.$$

By Lemma 2.1 we have that $Q_n(w,\lfloor nt/\tau\rfloor)$ converges in law with respect to the local uniform topology on $\mathbb{C}\times[0,\tau]$ (see Section 5) to $Q(w/\tau,t)$. Since Z is a deterministic transform, we also have that $F_n(w,t)$ converges in law to $F(w/\tau,t)$. We first show that μ_n is determined by F_n while μ is determined by F.

Recall that we defined

$$Q_{n,E}(w,\ell) = T^{-\ell}(E)M_{n,E}(w,\ell),$$

and so

$$\frac{2}{\rho(E)}\left(M_{n,E}(w,\lfloor nt/\tau\rfloor)\right)_{11} = \begin{pmatrix}1 & 0\end{pmatrix}\left(\frac{2}{\rho(E)}Z\right)D^{\lfloor nt/\tau\rfloor}Z^{-1}Q_{n,E}(w,\lfloor nt/\tau\rfloor)\begin{pmatrix}1\\0\end{pmatrix} \tag{3.4}$$

$$= z^{\lfloor nt/\tau\rfloor-1}F_n^1(w,t) + \overline{z}^{\lfloor nt/\tau\rfloor-1}F_n^2(w,t),. \tag{3.5}$$

{evector_transfor

In other words $\boldsymbol{m}_n^\lambda$ is a function of F_n. Moreover, for $\lambda\in\mathbb{R}$, we have by Equation (3.1) that $2\left|q^\lambda(t)\right|^2 = \left|F^1(\lambda,t)\right|^2 + \left|F^2(\lambda,t)\right|^2$ and so $\boldsymbol{q}^\lambda$ is a function of F.

Moreover, $\Lambda_{n,E} = \{w \in \mathbb{R} : m_n(w,\tau) = 0\}$, which again is determined by F_n. And in fact, $(2/\rho)m_{n_j}(w,\tau)$ converges in law to

$$\begin{aligned}\tilde{m}(w) &:= \lim_{n_j\to\infty} z^{n_j-1}F^1_{n_j}(w,t) + \overline{z}^{n_j-1}F^2_n(w,t)\\ &:= \tilde{z}\overline{z}F^1(w/\tau,\tau) + \overline{\tilde{z}}zF^2(w/\tau,\tau).\end{aligned}$$

And now notice that for $\lambda \in \mathbb{R}$, by Equation (3.1)

$$\tilde{m}(\lambda,\tau) = 0 \iff \tilde{z}\overline{z}iq(\lambda/\tau,\tau) + \overline{\tilde{z}\overline{z}iq(\lambda/\tau,\tau)} = 0 \iff \arg q(\lambda/\tau,\tau) + \arg(\tilde{z}-z) + \frac{\pi}{2} = 0.$$

In other words Sch^ϕ_τ is the zero set of $\tilde{m}$, which is determined by F.

We have shown that $\int h d\mu_n$ is a measurable function of F_n while $\int h d\mu$ is a measurable function of F. Since F_n converges in law to F, the continuous mapping theorem (eg. [Kal02], Theorem 3.27) allows us to remove the randomness from the problem. We may assume that F_n converges to F in the local uniform topology and simply show that this implies that $\int h d\mu_{n_j}$ converges to $\int h d\mu$. We may also assume that $h = h_1 \cdot h_2$, with $h_1 \in C_c(\mathbb{C})$ and $h_2 \in C(\mathcal{M}[0,\tau])$,

First notice that if $\lambda_n \to \lambda \in \mathbb{R}$, then as measures on $[0,\tau]$, $\boldsymbol{m}^{\lambda_n}_n$ converges weakly to $\boldsymbol{q}^{\lambda/\tau}$ (and so $h_2(\boldsymbol{m}^{\lambda_n}_n)$ converges to $h_2(\boldsymbol{q}^{\lambda/\tau})$). Take $u \in C[0,\tau]$, then

$$\int u\, d\boldsymbol{m}^{\lambda_n}_n = \int_0^\tau u(t) \left| z^{\lfloor nt/\tau \rfloor} F^1_n(\lambda_n,t) + \overline{z}^{\lfloor nt/\tau \rfloor} F^2_n(\lambda_n,t)\right|^2 dt.$$

Expanding the absolute value, noting that $F_n(\lambda_n,t)$ converge uniformly on $[0,\tau]$ to $F(\lambda/\tau,t)$, and applying Lemma (7.1) gives that

$$\begin{aligned}\lim_n \int u\, d\boldsymbol{m}^{\lambda_n}_n &= \int_0^\tau u(t)\left(\left|F^1(\lambda/\tau,t)\right|^2 + \left|F^2(\lambda/\tau,t)\right|^2\right) dt\\ &= \int_0^\tau u(t)\, d\boldsymbol{q}^{\lambda/\tau}(t).\end{aligned}$$

Moreover since F_n converges to F and z^{n_j} converges to $\tilde{z}$, the analytic functions on $\mathbb{C}$, $m_{n_j}(w,\tau)$ converge in the local uniform topology to $\tilde{m}(w/\tau)$. By Hurwitz's theorem this gives that the zeros of these functions converge pointwise. And the real valued zeros converge to real valued zeros. And so,

$$\lim_{n_j} \sum_{\lambda\in\mathbb{R}: m_{n_j}(\lambda,\tau)=0} h_1(\lambda)h_2(\boldsymbol{m}^{\lambda}_{n_j}) = \sum_{\lambda\in\mathbb{R}:\tilde{m}(\lambda/\tau)=0} h_1(\lambda)h_2(\boldsymbol{q}^{\lambda/\tau}),$$

which completes the proof. □

Proof of Lemma 3.5. Recall that $r^\lambda = \ln\left|q^\lambda\right|^2$. It therefore suffices to show that

$$\left\{\left(\lambda+u, r^{\lambda/\tau}\right) : \lambda \in \mathsf{Sch}^\phi_\tau\right\} =^d \left\{\left(\lambda, r^{\lambda/\tau}\right) : \lambda \in \mathsf{Sch}^*_\tau\right\}$$

We first show that for $u \in \mathbb{R}$ fixed,

$$\left\{\lambda+u, r^{\lambda/\tau}\right\}_{\lambda\in\mathsf{Sch}^\phi_\tau} =^d \left\{\lambda, r^{\lambda/\tau}\right\}_{\lambda\in\mathsf{Sch}^{\phi+u}_\tau}$$

Recall the SDEs from Lemma 3.3,

{SDE1} $$d\theta^\lambda = \lambda dt + d\mathcal{B} + \mathrm{Im}\left[e^{-i\theta^\lambda(t)} d\mathcal{W}\right], \quad \theta^\lambda(0) = 0 \tag{3.6}$$

{SDE2} $$dr^\lambda = \frac{dt}{4} + \mathrm{Re}\left[e^{-i\theta^\lambda(t)} d\mathcal{W}\right], \quad r^\lambda(0) = 0. \tag{3.7}$$

coupled together for all values of $\lambda \in \mathbb{R}$ where $\mathcal{B}$ and $\mathcal{W}$ are standard real and complex Brownian motions. We let $\tilde{\theta}^\lambda(t) := \theta^{\lambda - u/\tau}(t) + (u/\tau)t$ and $\tilde{r}^\lambda(t) := r^{\lambda-u/\tau}(t)$ and notice that $\tilde{\theta}^\lambda$ and $\tilde{r}^\lambda$ jointly solve Equations (3.6) and (3.7).

And so, since $\theta^{(\lambda-u)/\tau}(\tau) = \tilde{\theta}^{\lambda/\tau}(\tau) - u$,

$$\begin{aligned}\mathsf{Sch}_\tau^\phi + u &= \{\lambda : \theta^{(\lambda-u)/\tau}(\tau) \in 2\pi\mathbb{Z} + \phi\} \\ &= \{\lambda : \tilde{\theta}^{\lambda/\tau}(\tau) - u \in 2\pi\mathbb{Z} + \phi\}.\end{aligned}$$

Therefore

$$\begin{aligned}\left\{\left(\lambda + u, r^{\lambda/\tau}\right) : \lambda \in \mathsf{Sch}_\tau^\phi\right\} &= \left\{\left(\lambda, r^{(\lambda-u)/\tau}\right) : \lambda \in \mathsf{Sch}_\tau^\phi + u\right\} \\ &= \left\{\left(\lambda, \tilde{r}^{\lambda/\tau}\right) : \tilde{\theta}^{\lambda/\tau}(\tau) \in 2\pi\mathbb{Z} + \phi + u\right\} \\ &=^d \left\{\left(\lambda, r^{\lambda/\tau}\right) : \lambda \in \mathsf{Sch}_\tau^{\phi+u}\right\}\end{aligned}$$

by the uniqueness of solutions. Now if u is uniform on $[0, 2\pi]$, then $u + \phi \mod 2\pi$ is still uniform on $[0, 2\pi]$ and so $\mathsf{Sch}_\tau^{\phi+u} =^d \mathsf{Sch}_\tau^*$ which finishes the proof. □

Proof of Lemma 3.6. Recall that $\mathsf{Sch}_\tau^* = \{\lambda : \theta^{\lambda/\tau}(\tau) \in 2\pi\mathbb{Z} + v\}$, where v is uniform on $[0, 2\pi]$. Integrate out v to get

$$\begin{aligned}\mathbf{E} \sum_{\lambda \in \mathsf{Sch}_\tau^*} G(\lambda, r^{\lambda/\tau}) &= \frac{1}{2\pi} \mathbf{E} \int_0^{2\pi} du \sum_{\lambda : \theta^{\lambda/\tau}(\tau) \in 2\pi\mathbb{Z}+u} G(\lambda, r^{\lambda/\tau}) \\ &= \frac{1}{2\pi} \mathbf{E} \int_{-\infty}^{\infty} du \sum_{\lambda : \theta^{\lambda/\tau}(\tau) = u} G(\lambda, r^{\lambda/\tau}).\end{aligned}$$

Now using Lemma 3.3 we have that $\theta^{\lambda/\tau}(\tau)$ is almost surely a real analytic function in λ and $r^{\lambda/\tau}$ is continuous in λ so we can apply the co-area formula and then Fubini to get

$$\frac{1}{2\pi} \mathbf{E} \int_{-\infty}^{\infty} du \sum_{\lambda : \theta^{\lambda/\tau}(\tau) = u} G(\lambda, r^\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \mathbf{E}\left[G(\lambda, r^{\lambda/\tau}) \left|\frac{\partial \theta^{\lambda/\tau}(\tau)}{\partial \lambda}\right|\right] \tag{3.8}$$ {coarea}

From Lemma 3.3, we have that the evolution of r^λ is given by

$$dr^\lambda(t) = \frac{dt}{4} + \mathrm{Re}(e^{-i\theta^\lambda(t)} d\mathcal{W}).$$

And moreover, $\phi^{\lambda/\tau}(t) = \frac{\partial \theta^{\lambda/\tau}(t)}{\partial \lambda}$ is well defined, with SDE

$$d\phi^{\lambda/\tau} = \frac{dt}{\tau} - \mathrm{Re}(e^{-i\theta^{\lambda/\tau}} d\mathcal{W}) \phi^{\lambda/\tau}$$

Now fix λ and notice that $e^{-i\theta^\lambda} d\mathcal{W} =^d d\mathcal{W}$ and so r^λ and ϕ^λ do not depend on λ. We drop the λ dependence and jointly solve for r and ϕ to get

$$r_t = \frac{t}{4} + \frac{\mathcal{B}_t}{\sqrt{2}}$$
$$\phi_t = \frac{1}{\tau}\int_0^t du e^{(r_u - r_t)}.$$

And so by Fubini,

$$\mathbf{E}\left[G(\lambda, r^{\lambda/\tau})\left|\frac{\partial\theta^{\lambda/\tau}(\tau)}{\partial\lambda}\right|\right] = \frac{1}{\tau}\int_0^\tau du\mathbf{E}\left[e^{(r_u - r_\tau)}G(\lambda, r)\right],$$

Fix $u \in [0,\tau]$ and for simplicity, consider the process $\tilde{r}_t = \mathcal{B}_t + t/2$. This is just the time change $t \to 2t$. We will calculate the distribution of the path $\tilde{r}$ on $[0,\tau]$ weighted by $\exp(\tilde{r}_u - \tilde{r}_\tau)$. In other words if we take $\mathcal{R}$ to be the law of $\tilde{r}$ on $C[0,\tau]$, we need to characterize the measure on $C[0,\tau]$ given by,

$$\exp(\omega_u - \omega_\tau)\mathrm{d}\mathcal{R}(\omega).$$

By standard Girsanov theory, if we take $\mathcal{P}$ to be the law of Brownian motion on $C[0,\tau]$, then $\mathrm{d}\mathcal{R}(\omega) = \exp\left(\frac{\omega_\tau}{2} - \frac{\tau}{8}\right)d\mathcal{P}(\omega)$ and so

$$\exp(\omega_u - \omega_\tau)d\mathcal{R}(\omega) = \exp\left(\omega_u - \frac{\omega_\tau}{2} - \frac{\tau}{8}\right)d\mathcal{P}(\omega). \tag{3.9}$$

Now if we let $x^u := x^u(\omega)$ be the Brownian path reflected at u, we have that the corresponding exponential martingale of $x^u/2$ at τ is

$$\exp\left(\frac{x^u_\tau}{2} - \frac{[x^u]_\tau}{8}\right) = \exp\left(\omega_u - \frac{\omega_\tau}{2} - \frac{\tau}{8}\right)$$

where $[x^u]_t$ is the quadratic variation of x^u at t. Therefore, by another application of Girsanov, if we let $f^u_t = [x^u/2, \omega]_t = \frac{1}{2}(u - |u-t|)$, then under the measure $\exp(\omega_u - \omega_\tau)d\mathcal{R}(\omega)$ on $C[0,\tau]$ a path ω is distributed like $\mathcal{B} + f^u$ where $\mathcal{B}$ is a standard Brownian motion. Undoing the time change and applying Brownian scaling gives that,

$$\mathbf{E}\left[e^{(r_u - r_\tau)}G(\lambda, r)\right] = \mathbf{E}\left[G\left(\lambda, \frac{\mathcal{B}}{\sqrt{2}} + \frac{f^u}{2}\right)\right],$$

which completes the proof.

□

Proof of Lemma 3.3. We let $X(\lambda,t) = Z^{-1}Q(\lambda,t)$. From Equation (2.8) we have the following stochastic differential equation for X in t,

$$dX(\lambda,t) = \frac{1}{2}\left(\begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix}dt + \begin{pmatrix} id\mathcal{B} & d\mathcal{W} \\ d\overline{\mathcal{W}} & -id\mathcal{B}\end{pmatrix}\right)X(\lambda,t), \qquad X(\lambda,0) = Z^{-1}.$$

This gives that

$$dX_{11}(\lambda,t) = \frac{i\lambda}{2}X_{11}(\lambda,t)\,dt + iX_{11}(\lambda,t)d\mathcal{B} + X_{21}(\lambda,t)d\mathcal{W}.$$

If $\lambda \in \mathbb{R}$, then $X(\lambda,t)_{11} = \overline{X(\lambda,t)_{21}}$ and moreover $q^\lambda(t) = iX(\lambda,t)_{11}$. We fix $\lambda \in \mathbb{R}$ and drop it from our notation to get

$$dq = \frac{i\lambda}{2} q\, dt + \frac{1}{2}\left(iq d\mathcal{B} - \overline{q} d\mathcal{W}\right) \quad q(0) = 1$$

Ito's formula then gives that

$$\begin{aligned} d\log q &= \frac{dq}{q} - \frac{1}{2}\frac{(dq)^2}{q^2} \\ &= \frac{i\lambda}{2}dt + \frac{i}{2}d\mathcal{B} + \frac{1}{2}\frac{\overline{q}}{q}d\mathcal{W} + \frac{dt}{8} \end{aligned}$$

Since $r = 2\mathrm{Re}\ln q$ and $\theta = 2\mathrm{Im}\ln q$, this yields for $\lambda \in \mathbb{R}$, the following SDEs in t,

$$\begin{aligned} dr &= \mathrm{Re}\left(\frac{\overline{q}}{q}d\mathcal{W}\right) + \frac{dt}{4}, \\ d\theta &= \lambda dt + d\mathcal{B} + \mathrm{Im}\left(\frac{\overline{q}}{q}d\mathcal{W}\right). \end{aligned}$$

Noting that $\frac{\overline{q}}{q} = \exp(-i\theta)$ finishes the proof. $\square$

4. Proof of Theorem 1.1

{LocalImpliesGloba

We are now in a position to prove the main theorem of the paper. We will average the local result of Theorem 3.1 to get the more macroscopic version of the theorem. In order to do so we need to be able to control the number of eigenvalues in an a microscopic interval (of size $1/(\rho n)$) around E. We will the need the following lemma whose proof is given in Section 6.

{mom_num_evalues}

Lemma 4.1. *Fix $R > 0$ and let $\Delta_n(E) = \left(E - \frac{R}{n\rho(E)}, E + \frac{R}{n\rho(E)}\right)$. Furthermore, let $N_n(E) = |\Lambda_n \cap \Delta_n(E)|$ be the number of eigenvalues of H_n in $\Delta_n(E)$. Then for any $\epsilon > 0$,*

$$\sup_n \sup_{E \in (-2+\epsilon, 2-\epsilon)} \mathbf{E}\left[N_n(E)\right]^{3/2} < \infty.$$

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1. Take θ uniform on $[0,2\pi]$ and let $\boldsymbol{\psi}_n^\mu \in \mathcal{M}[0,1]$ with density $\left|\psi^\mu\left(\lfloor nt \rfloor\right)\right|^2 dt$. Using Theorem 3.1 and the time change $t \to \tau t$, we have that for $0 < |E| < 2$, the point process

$$\mathcal{P}_{E,n} = \left\{\left(n\rho(E)(\mu - E) + \theta, n\boldsymbol{\psi}_n^\mu\right) : \mu \in \Lambda_n\right\}.$$

converges in law to a limiting point process $\mathcal{P}_\tau$.
In particular, if we fix $g_1 = (1-|x|)\mathbf{1}_{[|x|\le 1]}$, $g_2 \in C_b(\mathbb{R} \times \mathcal{M}[0,1])$ and let

$$G_n(E) := \sum_{\mu \in \Lambda_n} g_1\Big(n\rho(E)(\mu - E)\Big) g_2\left(\mu, \boldsymbol{\psi}_n^\mu\right).$$

Then for fixed $|E| < 2$, $G_n(E)$ converges in distribution to $G(E)$ and

$$\mathbf{E}G(E) = \frac{1}{2\pi}\mathbf{E}g_2\left(E, \frac{S(\tau(t-u))dt}{\int_0^1 ds\, S(\tau(s-u))}\right). \tag{4.1}$$

{limit_expect}

We now show that $\int \mathbf{E}G_n(E) d\rho(E)$ converges to $\int \mathbf{E}G(E)d\rho(E)$ from which the result will follow.

Fix $\epsilon > 0$. Since $\operatorname{supp} g_1 \subset [-1,1]$, we let

$$N_n(E) = \{\mu \in \Lambda_n : |\mu - E| \le 1/(n\rho(E)\},$$

which gives that $G_n(E) \le \|g_1\|_\infty \|g_2\|_\infty N_n(E)$. And so from Theorem 4.1,

$$\sup_n \sup_{0<|E|<2-\epsilon} \mathbf{E}\,[G_n(E)]^{3/2} < \infty.$$

Therefore $G_n(E)\mathbf{1}_{|E|<2-\epsilon}$ is uniformly integrable with respect to $\mathbf{P} \times d\rho$. And so since $G_n(E)$ converges in law to $G(E)$, we have that

vergence_epsilon}

$$\lim_{n\to\infty} \int d\rho(E)\mathbf{E}\Big[G_n(E)\mathbf{1}_{[|E|<2-\epsilon]}\Big] = \int d\rho(E)\mathbf{E}\Big[G(E)\mathbf{1}_{[|E|<2-\epsilon]}\Big]. \tag{4.2}$$

Now by Fubini,

$$\int d\rho(E)\mathbf{E}\Big[G_n(E)\mathbf{1}_{[|E|<2-\epsilon]}\Big] = \mathbf{E}\sum_{\mu\in\Lambda_n} g_2\left(\mu, \boldsymbol{\psi}_n^\mu\right) \int_{-2+\epsilon}^{2-\epsilon} d\rho(E) g_1\Big(n\rho(E)(\mu - E)\Big).$$

Fix $\delta > \epsilon$ and let $A_n(\delta) = \{\mu \in \Lambda_n : |\mu| < 2-\delta\}$, $B_n(\delta) = \{\mu \in \Lambda_n : |\mu| \ge 2-\delta\}$. We write

$$\int d\rho(E)\mathbf{E}\Big[G_n(E)\mathbf{1}_{[|E|<2-\epsilon]}\Big] = \mathbf{E}\left[\sum_{\mu\in A_n(\delta)} g(\mu)\right] + \mathbf{E}\left[\sum_{\mu\in B_n(\delta)} g(\mu)\right],$$

with

$$g(\mu) = g_2\left(\mu, \boldsymbol{\psi}_n^\mu\right) \int_{-2+\epsilon}^{2-\epsilon} d\rho(E) g_1\Big(n\rho(E)(\mu - E)\Big),$$

and deal with each piece separately.

First notice that for $k \in \mathbb{N}$, we can bound

$$\begin{aligned} |B_n(\delta)| &\le \sum_{\mu\in B_n(\delta)} \left(\frac{\mu}{2-\delta}\right)^{2k} \\ &\le (2-\delta)^{-2k} \sum_{\mu\in\Lambda_n} \mu^{2k}, \end{aligned}$$

We know (see ??) that for fixed k,

$$\begin{aligned} \lim_{n\to\infty} \mathbf{E}\left[\frac{1}{n}\sum_{\mu\in\Lambda_n} \mu^{2k}\right] &= \frac{1}{2\pi}\int x^{2k}\rho(x)dx \\ &\le C\frac{2^{2k}}{\sqrt{k}}. \end{aligned}$$

Taking $k = \lfloor 1/\delta \rfloor$, we have that $\big(1-(\delta/2)\big)^{-2k}$ is bounded independent of δ. And so,

$$\lim_{n\to\infty} \frac{1}{n}\mathbf{E}\,|B_n(\delta)| \le C\frac{\big(1-(\delta/2)\big)^{-2k}}{\sqrt{k}} \tag{4.3}$$

{Bdelta_bound}

$$\le C\sqrt{\delta} \tag{4.4}$$

Now use the second part of Lemma 7.2 to get that for $\mu \in B_n(\delta)$,

$$\int d\rho(E) g_1\Big(n\rho(E)(\mu - E)\Big) \leq \frac{D}{n}.$$

And along with equation (4.4) this gives

$$\begin{aligned}\mathbf{E} \sum_{\mu \in B_n(\delta)} g(\mu) &\leq \|g_2\|_\infty \frac{D}{n} |B_n(\delta)| \\ &= O(\sqrt{\delta}).\end{aligned}$$

Now for n large enough if $\mu \in A_n(\delta)$,

$$\begin{aligned}\int_{-2+\epsilon}^{2-\epsilon} g_1\Big(n\rho(x)(x-\mu)\Big) d\rho(x) &= \int_{-2}^{2} g_1\Big(n\rho(x)(x-\mu)\Big) d\rho(x) \\ &= \frac{1}{n}\int g_1(x)\, dx + o\left(1/n\right) \\ &= \frac{1}{n} + o\left(1/n\right)\end{aligned}$$

The first equality follows from the fact that for $x \in [-2, 2]$, $\rho(x) \geq 1$. And so since $g_1 \in C_c(\mathbb{R})$, we have that $|x - \mu| \leq D/n$ for some constant D. Since $\mu < 2 - \delta$, we have that $|x| < 2 - \epsilon$ for n large enough. The second equality follows from Lemma 7.2. And so

$$\begin{aligned}\mathbf{E} \sum_{\mu \in A_n(\delta)} g(\mu) &= \frac{1}{n} \sum_{\mu \in A_n(\delta)} \mathbf{E} g_2\big(\mu, \boldsymbol{\psi}^\mu\big) + o(1) \\ &= \frac{1}{n} \sum_{\mu \in \Lambda_n} \mathbf{E} g_2\big(\mu, \boldsymbol{\psi}^\mu\big) + O(\sqrt{\delta}) + o(1),\end{aligned}$$

with the last equality coming from equation (4.4). To sum up

$$\int d\rho(E)\mathbf{E}\Big[G_n(E)\mathbf{1}_{[|E|<2-\epsilon]}\Big] = \frac{1}{n} \sum_{\mu \in \Lambda_n} \mathbf{E} g_2(\mu, \boldsymbol{\psi}^\mu) + o(1) + O(\sqrt{\delta}). \tag{4.5}$$ {average_average}

On the other hand,

$$\int d\rho(E)\mathbf{E}\Big[G(E)\mathbf{1}_{[|E|<2-\epsilon]}\Big] = \int d\rho(E)\mathbf{E}\left[G(E)\right] + O(\epsilon).$$

And so by equation (4.1) along with equation (4.5) and the convergence from equation (4.2) we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\mu \in \Lambda_n} \mathbf{E}\, g_2\big(\mu, \boldsymbol{\psi}^\mu\big) = \frac{1}{2\pi} \int d\rho(E) \mathbf{E} g_2 \left(E, \frac{S(\tau(t-u))dt}{\int_0^1 ds\, S(\tau(s-u))}\right) + O(\epsilon) + O(\delta).$$

Since $\delta > \epsilon$ was arbitrary, this completes the proof. $\square$

5. Tightness

{Tightness}

In this section we discuss the underlying tightness bounds we need to prove the weak convergence in Lemma 2.1.

We will use the following notions of convergence. Let $\mathcal{A}_d$ denote the space of continuous functions from $\mathbb{C} \times [0,1]$ to $\mathbb{C}^d$ that are also analytic in the first variable. In other words, if $f \in \mathcal{A}_d$, then for every $t \in [0,1]$, $f(\cdot, t)$ is an analytic function from $\mathbb{C}$ to $\mathbb{C}^d$. We equip $\mathcal{A}_d$ with the metric

$$d(f,g) := \sum_{r=1}^{\infty} 2^{-r} \frac{\|f-g\|_r}{1+\|f-g\|_r}, \qquad \|h\|_r := \max_{(x,z)\in D_r} \|h(z,x)\|,$$

where $D_r = B_r \times [0,1]$ and $B_r = \{w \in \mathbb{C} : |w| \le r\}$. Under this metric $\mathcal{A}_d \subset C\left([0,1]\times\mathbb{C}, \mathbb{C}^d\right)$ is a complete, separable metric space.

A random function in $\mathcal{A}_d$ is a measurable mapping $\omega \to f \in \mathcal{A}_d$ from a probability space $(\Omega, \mathcal{F}, P)$ to $(\mathcal{A}_d, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-field generated by the metric d. The law of f is the induced probability measure ρ_f on $(\mathcal{A}_d, \mathcal{B}_d)$. A sequence f_ℓ of random analytic functions is said to converge in law to a random $f \in \mathcal{A}_d$ if $\rho_{f_\ell} \to \rho_f$ in the usual sense of weak convergence.

alytic_functions}

Proposition 5.1. *Suppose f_ℓ is a sequence of random functions in $\mathcal{A}_d$ such that*
(1) *For every $w \in \mathbb{C}$, the processes $f_\ell(w,\cdot) \in C\left([0,1],\mathbb{C}^d\right)$ are tight,*
(2) *For every $r > 0$,*

{bound_tight}

$$\lim_{M\to\infty} \sup_{\ell} \mathbf{P}\left(\|f_\ell\|_r > M\right) = 0, \tag{5.1}$$

(3) *For each $m \ge 1$ and $(z,t) = ((z_1,t_1),(z_2,t_2),\cdots,(z_m,t_m)) \in (\mathbb{C}\times[0,1])^m$ there is a probability distribution $\nu_m^{(z,t)}$ on $(\mathbb{C}^d)^m$ and the random vector $(f_\ell(z_1,t_1), f_\ell(z_2,t_2),\cdots,f_\ell(z_m,t_m)) \in \left(\mathbb{C}^d\right)^m$ converges in law to $\nu_m^{z,t}$.*
Then there is a random function f in $\mathcal{A}_d$ such that f_ℓ converges in law to f. Moreover for each $(z,t) = ((z_1,t_1),(z_2,t_2),\cdots,(z_m,t_m)) \in (\mathbb{C}\times[0,1])^m$, $(f(z_1,t_1), f(z_2,t_2),\cdots,f(z_m,t_m)) \in \mathbb{C}^m$ has distribution $\nu_m^{(z,t)}$.

Proof. We first show that Assumptions (1) and (2) imply that the sequence f_ℓ is tight. We may assume that each $f_\ell \in \mathcal{A}_1$ since tightness in every coordinate function implies that the sequence is tight.
Fix $r > 0$, $|w|$, $|u| \le r$, and take $f \in \mathcal{A}_1$. Then, by Cauchy's integral formula,

$$\begin{aligned} f(w,t) - f(u,t) &= C_r \int_{|z|=2r} \left(\frac{f(z,t)}{w-z} - \frac{f(z,t)}{u-z} \right) dz \\ &= C_r \int_{|z|=2r} \frac{f(z,t)}{(w-z)(u-z)} (u-w)\, dz \end{aligned}$$

And so Jensen's inequality along with the fact that $|z-u|$, $|w-u| \ge r$ gives that, for every t,

$$|f(w,t) - f(u,t)| \le C_r \|f\|_{2r} |u-w|.$$

This inequality gives that for $|\zeta| \leq r$,

$$|f(u,t) - f(w,s)| \leq C_r \|f\|_{2r} (|u-\zeta| + |w-\zeta|) + |f(\zeta,t) - f(\zeta,s)| .$$

And so if we take any α-net $K_\alpha \subset B_r$ and take $\delta < \alpha/2$,

$$\sup_{\substack{\|(w,t)-(u,s)\|<\delta \\ |w|,|u|\leq r}} |f(w,t) - f(u,s)| \leq 2C_r \|f\|_{2r} \alpha + \max_{w\in K_\alpha} \sup_{|s-t|<\delta} |f(w,t) - f(w,s)| . \tag{5.2}$$ {net_ineq}

Now fix $\epsilon > 0$. Since $f_\ell(w,\cdot)$ is tight for $w \in \mathbb{C}$, for every $\gamma > 0$ we can find a $\delta_w > 0$ such that

$$\sup_{\ell\in\mathbb{N}} \mathbb{P}\left(\sup_{|s-t|<\delta} |f_\ell(w,t) - f_\ell(w,s)| > \epsilon \right) < \gamma.$$

In fact, just by adding probabilities, for any $\gamma, \alpha > 0$ we can find a finite α-net $K_\alpha \subset B_r$ and a $\delta_\alpha > 0$ such that,

$$\sup_{\ell\in\mathbb{N}} \mathbb{P}\left(\max_{w\in K_\alpha} \sup_{|s-t|<\delta_\alpha} |f_\ell(w,t) - f_\ell(w,s)| > \epsilon \right) < \gamma. \tag{5.3}$$ {dense_tight}

Now fix $\gamma > 0$. Assumption (2) means that we can find an M such that $\mathbb{P}(||f_\ell||_{2r} > M) < \gamma$. Take $\alpha < \epsilon(2MC_r)^{-1}$ and find a finite α-net K_α and a δ_α satisfying Equation (5.3). Finally take $\delta = \min(\delta_\alpha, \alpha/2)$. Using Equation (5.2), we get that,

$$\sup_{\ell\in\mathbb{N}} \mathbb{P}\left(\sup_{\substack{\|(w,t)-(u,s)\|<\delta \\ |w|,|u|\leq r}} |f_\ell(w,t) - f_\ell(u,s)| \geq 2\epsilon \right) < 2\gamma. \tag{5.4}$$

Since ϵ and γ were arbitrary, this inequality along with Assumption (2) and Arzelà-Ascoli gives tightness of the sequence f_ℓ restricted to the discs D_r. And so by Prokohorov's theorem a subsequence of f_ℓ restricted to D_r converges in law. By a diagonal argument, there is a subsequence f_{ℓ_k} such that for each integer r, the restriction of f_{ℓ_k} to D_r converges to a random analytic function f_r on D_r. The distributions of the functions f_r are consistent with respect to restricting to smaller discs, and thus there is a random analytic function f on $\mathbb{C} \times [0,1]$ such that $f_{\ell_k} \to f$ in law with respect to the local uniform topology. Condition (2) is strong enough to ensure that f is unique and thus $f_\ell \to f$ in law.

□

Proof of Theorem 2.1. We intend to apply Lemma 5.1 to $Q_n(w,t) := Q_{n,E}(w, \lfloor nt/\tau \rfloor)$. We cannot apply this directly since for any $w \in \mathbb{C}$, the processes $Q_n(w,\cdot)$ are piecewise constant but not continuous. Instead, for all $w \in \mathbb{C}$ we let $\tilde{Q}_n(w,\cdot)$ be the linearized version of the process $Q_n(w,\cdot)$. By this we mean the function whose graph is given by the straight line between each consecutive jump discontinuity of $Q_n(w,\cdot)$. Since Q_n are analytic for any fixed t, $\tilde{Q}_n \in \mathcal{A}_4$. Theorem 1 of [KVV12] gives the tightness bound (2) for $\tilde{Q}_n$. Theorem 2 of [KVV12] and the continuous mapping theorem gives that for fixed $w \in \mathbb{C}$, $\tilde{Q}_n(w,\cdot)$ converge in law with respect to the uniform topology and so by by Prokhorov the tightness bound (1). This theorem also gives convergence of the finite dimensional distributions of Q_n and hence those of $\tilde{Q}_n$ which is condition (3). So by Lemma 5.1 Q_n

converges in law to Q and since $d(Q_n, \tilde{Q}_n)$ goes to zero in probability we get that Q_n converges in law to Q with respect to the local uniform topology.

$\square$

alEvalueEstimate}

6. Local Eigenvalue Estimate

In this section we give the proof of Lemma 4.1. The moment bound on the number of eigenvalues in a macroscopic interval follows from an application of Theorem 2.2 of [LS06].

Theorem 6.1 ([LS06])**.** *Let $\mu < \mu'$ be consecutive eigenvalues of H_n. Then for any $E \in (\mu, \mu')$,*

{evalue_space_lb}

$$\mu' - \mu \geq \left(\sum_{\ell=1}^{n} \|M_n(E,\ell)\|^2 \right)^{-1}. \tag{6.1}$$

_evalue_transfer}

Corollary 6.2. *For any interval $\Delta \subset \mathbb{R}$, let $N_n(\Delta) := |\Lambda_n \cap \Delta|$ be the number of eigenvalues of H_n in Δ. Then,*

$$N_n(\Delta) \leq 1 + |\Delta|^2 \int_\Delta dE \left(\sum_{\ell=1}^{n} \|M_n(E,\ell)\|^2 \right).$$

Proof. Fix $n \in \mathbb{N}$ and let $\tau(E) := \sum_{\ell=1}^{n} \|M_n(E,\ell)\|^2$. Take $\mu < \mu' \in \Delta$ consecutive eigenvalues of H_n. Integrating equation (6.1) gives

$$\begin{aligned}(\mu' - \mu) &\geq \frac{1}{\mu' - \mu} \int_{\mu}^{\mu'} \frac{dE}{\tau(E)} \\ &\geq \frac{1}{|\Delta|} \int_\Delta \frac{dE}{\tau(E)}.\end{aligned}$$

This gives a uniform lower bound on the distance between any two consecutive eigenvalues in Δ. And so by Jensen's inequality,

$$\begin{aligned}N_n(\Delta) &\leq 1 + \left(|\Delta| \Big/ \frac{1}{|\Delta|} \int_\Delta \frac{dE}{\tau(E)} \right) \\ &\leq 1 + |\Delta|^2 \int_\Delta dE \tau(E).\end{aligned}$$

$\square$

To prove Theorem 4.1 via Corollary 6.2 we need a moment bound on the transfer matrices.

rm_process_bound}

Lemma 6.3. *Let $\|\cdot\|$ be the Hilbert-Schmidt norm on $M_{2\times 2}(\mathbb{C})$. There is a continuous function f on $(-2,2)$ such for every $E \in (-2,2)$,*

$$\sup_n \max_{0 \leq \ell \leq n} \mathbf{E} \|M_n(E,\ell) - I\|^3 < f(E).$$

Proof. Fix $E \in (-2,2)$ and $n \in \mathbb{N}$ and recall that for $0 \leq \ell \leq n$,

$$M_n(E,\ell) = T(E - v_{\ell,n}) T(E - v_{\ell-1,n}) \cdots T(E - v_{1,n}),$$

with $T(x) := \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$ and $v_{\ell,n} = \frac{\sigma\omega_\ell}{\sqrt{n}}$.
We will prove a bound for the process $X_\ell = T^{-\ell}(E)M_n(E,\ell)$. Using the identity

$$T(y)T^{-1}(x) = I + \begin{pmatrix} 0 & y-x \\ 0 & 0 \end{pmatrix},$$

we have that

$$X_\ell = T^{-\ell}T(E - v_{\ell,n})T^{-1}T^\ell X_{\ell-1} \tag{6.2}$$

$$= (I - v_{\ell,n}\mathcal{E}_\ell)X_{\ell-1}, \tag{6.3}$$

{mtx_recursion}

where $\mathcal{E}_\ell = T^{-\ell}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}T^\ell(E)$.
We first show that

$$\|\mathcal{E}_\ell\| \le c_1(\rho(E))^2, \tag{6.4}$$

{cE_bound}

where c_1 does not depend on n or E and $\rho(E) = 1/\sqrt{1-(E/2)^2}$. Recall that we can write $T(E) = ZDZ^{-1}$ where

$$D = \begin{pmatrix} \overline{z} & 0 \\ 0 & z \end{pmatrix}, \quad Z = \frac{i\rho(E)}{2}\begin{pmatrix} \overline{z} & z \\ 1 & 1 \end{pmatrix}, \quad Z^{-1} = \begin{pmatrix} 1 & -z \\ -1 & \overline{z} \end{pmatrix}. \tag{6.5}$$

with $z = E/2 + i\sqrt{1-(E/2)^2}$.
Using the submultiplicativity of the Hilbert-Schmidt norm along with the fact that $|z| = 1$ gives that for every $\ell \in \mathbb{Z}$,

$$\left\|T^\ell(E)\right\| \le 16\rho(E).$$

And since $\|\mathcal{E}_\ell\| \le \left\|T^\ell(E)\right\|\left\|T^{-\ell}(E)\right\|$, we get the bound (6.4).

Now notice that X_ℓ is a martingale with $X_0 = I$. We use the Burkholder-Davis-Gundy inequality along with Doob's Decomposition to get that for $0 \le \ell \le n$,

$$\mathbf{E}\max_{k\le\ell}\|X_k - I\|^3 \le c_2\,\mathbf{E}\left(\sum_{k=1}^{\ell}\mathbf{E}\left[\|X_k - X_{k-1}\|^2|\mathcal{F}_{k-1}\right]\right)^{3/2},$$

Now use that $X_k - X_{k-1} = v_k\mathcal{E}_kX_{k-1}$, the bound on $\mathcal{E}_k$, and that $\mathbf{E}v_{\ell,n}^2 = \sigma^2/n$ to get that

$$\mathbf{E}\max_{k\le\ell}\|X_k - I\|^3 = c_2\,\mathbf{E}\left(\frac{c_1\sigma^2\rho(E)^2}{n}\sum_{k=1}^{\ell}\|X_{k-1}\|^2\right)^{3/2}$$

$$\le c_3\rho(E)^3\frac{1}{n}\,\mathbf{E}\sum_{k=1}^{\ell}\|X_{k-1}\|^3,$$

with the last inequality following from Jensen. Now using the inequality $\|A+B\|^p \leq 2^p(\|A\|^p + \|B\|^p)$,

$$\mathbf{E}\max_{k\leq \ell}\|X_k - I\|^3 \leq \frac{c_3\rho(E)^3}{n}\sum_{k=1}^{\ell}\left(\mathbf{E}\|X_{k-1}-I\|^3 + \|I\|^3\right) \tag{6.6}$$

$$\leq c_4\rho(E)^3\left(1+\frac{S_{\ell-1}}{n}\right), \tag{6.7}$$

{max_norm_bound}

where we have set $S_\ell = \sum_{k=1}^{\ell}\mathbf{E}\|X_k - I\|^3$. This gives that

$$\begin{aligned} S_\ell - S_{\ell-1} &= \mathbf{E}\|X_\ell - I\|^3 \\ &\leq c_4\rho(E)^3\left(1+\frac{S_{\ell-1}}{n}\right), \end{aligned}$$

Finally, letting $R_\ell = 1 + S_\ell/n$, we have that $R_\ell \leq R_{\ell-1}(1 + c_4\rho(E)^3/n)$, and so $R_\ell \leq \exp(c\rho(E)^3)$ for $1 \leq \ell \leq n$. Therefore, equation (6.7) gives that

$$\begin{aligned} \mathbf{E}\max_{0\leq k\leq n}\|X_k - I\|^3 &\leq c_4\rho(E)^3 R_{n-1} \\ &\leq d_1\rho(E)^3\exp(d_2\rho(E)^3), \end{aligned}$$

for some constants d_1 and d_2 that do not depend on E or n. Since $M_n(E,\ell) = T^{-\ell}(E)X_\ell$, this finishes the proof. □

Proof of Theorem 4.1. Using Corollary 6.2 we have that

$$|N_n(E)-1|^{3/2} \leq \max\left(\left[|\Delta_n(E)|^2\int_{\Delta_n(E)}dx\sum_{\ell=1}^{n}\|M_n(E,\ell)\|^2\right]^{3/2},\ 1\right).$$

Since $|\Delta_n(E)| = 2R/(\rho(E)n$, we apply Jensen twice to get

$$\mathbf{E}\left[|\Delta_n(E)|^2\int_{\Delta_n(E)}dx\sum_{\ell=1}^{n}\|M_n(x,\ell)\|^2\right]^{3/2} \leq \frac{g(E)}{n^3}\mathbf{E}\int_{\Delta_n(E)}dx\sum_{\ell=1}^{n}\|M_n(x,\ell)\|^3.$$

Here g is continuous on $(-2,2)$. Now we use Fubini along with Lemma 6.3 to get that,

$$\mathbf{E}\int_{\Delta_n(E)}dx\sum_{\ell=1}^{n}\|M_n(x,\ell)\|^3 \leq \frac{R}{\rho(E)}sup_{x\in\Delta_n(E)}f(x).$$

Now fix $\epsilon > 0$ and $I_\epsilon = (-2+\epsilon, 2-\epsilon)$. There is an $N \in \mathbb{N}$ such that for any $n \geq N$ if $E \in I_\epsilon$, then $\Delta_n(E) \subset I_{\epsilon/2}$. Since f is continuous on $(-2,2)$ this means that for $n \geq N$,

$$\mathbf{E}|N_n(E)-1|^{3/2} \leq \max\left(\frac{C}{n^3}, 1\right)$$

□

{analytic_fact}

7. Appendix - some integrals

Theorem 7.1. *Let $D([0,1],\mathbb{C})$ be the space of cadlag functions from $[0,1]$ to $\mathbb{C}$. Suppose the sequence $f_n \in D([0,1],\mathbb{C})$ converges uniformly to $f \in C([0,1],\mathbb{C})$. Then for fixed $z \in \mathbb{C}$, $|z| = 1$ but $z \neq 1$,*

$$\lim_{n\to\infty} \int_0^1 f_n(t) z^{\lfloor nt \rfloor} dt = 0.$$

Proof. Since

$$\left| \int_0^1 f_n(t) z^{\lfloor nt \rfloor} dt - \int_0^1 f(t) z^{\lfloor nt \rfloor} dt \right| \leq \|f_n - f\|,$$

it suffices to show that for any continuous $f : [0,1] \to \mathbb{C}$,

$$\lim_{n\to\infty} \int_0^1 f(t) z^{\lfloor nt \rfloor} dt = 0.$$

We first assume that f is simple, by which we mean that $f := c\mathbf{1}_{[a,b)}$, for some constant c and subinterval $(a,b) \subset [0,1]$. We have that

$$\int_0^1 f(t) z^{\lfloor nt \rfloor} = \frac{c}{n} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} z^k + o\left(\frac{1}{n}\right).$$

Since $z \neq 1$, $\sum_{k=0}^N z^k$ is bounded for all $N \in \mathbb{N}$, which finishes this case. Additivity then gives the result for any finite sum of piecewise, simple functions. And for a general $f \in C([0,1],\mathbb{C})$, we can find functions g_m which are finite sums of simple functions so that

$$\sup_n \left| \int_0^1 g_m(t) z^{\lfloor nt \rfloor} dt - \int_0^1 f(t) z^{\lfloor nt \rfloor} dt \right| \leq \int_0^1 |g_m(t) - f(t)|\, dt < \epsilon_m,$$

with $\epsilon_m \to 0$. This completes the proof. □

{rho_integral}

Lemma 7.2. *Let $\rho(x) = 1/\sqrt{1-(x/2)^2}$. Fix $\epsilon > 0$ and $F \in C_c(\mathbb{R})$. Then*

$$\sup_{|\mu|<2-\epsilon} \left| \int F(n\rho(x)(\mu - x))\, \rho(x) dx - \int F(x) dx \right| = o\left(\frac{1}{n}\right)$$

Proof. Suppose that $\mathrm{supp} F \subset [-R,R]$ for some $R > 0$. Then we can suppose $|\mu - x| \leq R/n$ because otherwise since $\rho \geq 1$ we have that $F(n\rho(x)(\mu - x)) = F(n\rho(\mu)(\mu - x)) = 0$. ρ is Lipschitz on any closed subset of $(-2,2)$ and so for n large enough (depending only on ϵ) we have that

- $|\rho(\mu) - \rho(x)| \leq C/n$,
- $|n\rho(\mu)(\mu - x) - n\rho(x)(\mu - x)| \leq \frac{RC}{n}$.

This implies that

$$\begin{aligned} \int |F(n\rho(x)(\mu - x))\, \rho(x) dx - F(n\rho(x)(\mu - x))\, \rho(\mu) dx| &\leq \frac{C}{n} \int F(n\rho(x)(\mu - x))\, dx \\ &\leq \frac{CR\|F\|}{n^2}. \end{aligned}$$

And also that,

$$\begin{aligned}\int \left|F\left(n\rho(x)(\mu-x)\right)\rho(\mu)dx - F\left(n\rho(\mu)(\mu-x)\right)\rho(\mu)dx\right| &\le \rho(\mu) \sup_{|x-y|\le CR/n} |F(x)-F(y)| \int \mathbf{1}[|\mu-x|<R/n]dx \\ &\le \frac{D}{n} \sup_{|x-y|\le CR/n} |F(x)-F(y)| \\ &= o\left(1/n\right)\end{aligned}$$

since F is uniformly continuous. These two inequalities imply

$$\sup_{|\mu|<2-\epsilon} \left| \int F\left(n\rho(x)(\mu-x)\right)\rho(x)dx - \int F\left(n\rho(\mu)(\mu-x)\right)\rho(\mu)dx \right| = o\left(1/n\right).$$

And we are done since $\int F\left(n\rho(\mu)(\mu-x)\right)\rho(\mu)dx = \int F(x)dx$. □

Lemma 7.3. *Let $\rho(x) = 1/\sqrt{1-(x/2)^2}$ and take $F \in C_c(\mathbb{R})$ with $F \ge 0$ and $F(x) < F(y)$ for $|x| > |y|$. Then,*

$$\sup_{|\mu|<2} \int_{-2}^{2} F\left(n\rho(x)(x-\mu)\right)\rho(x)dx \le O\left(\frac{1}{n}\right).$$

Proof. By symmetry of $\rho(x)$, we can assume $\mu \ge 0$. Since $\rho(x) \ge 1$, we have that $|x-\mu| \le R/n$, where supp$F \subset [-R,R]$. In particular, since $\mu \ge 0$, for n large enough, we have that x is bounded away from -2 independently of μ. And so we can write

$$\frac{c_1}{\sqrt{2-x}} \le \rho(x) \le \frac{c_2}{\sqrt{2-x}}.$$

The decreasing property of F gives that

$$\int_{-2}^{2} F\left(n\rho(x)(x-\mu)\right)\rho(x)dx \le c_2 \int_{-2}^{2} F\left(c_1 n \frac{x-\mu}{\sqrt{2-x}}\right)\frac{dx}{\sqrt{2-x}}.$$

Writing $\gamma = 2-\mu$ and changing variables $y = \sqrt{2-x}/\sqrt{\gamma}$,

$$\begin{aligned}\int_{-2}^{2} F\left(c_1 n \frac{x-\mu}{\sqrt{2-x}}\right)\frac{dx}{\sqrt{2-x}}. &= \sqrt{\gamma}\int_0^{2/\sqrt{\gamma}} F\left(c_1 n\sqrt{\gamma}\left(\frac{1-y^2}{y}\right)\right)dy \\ &\le C\left\|F\right\|\sqrt{\gamma}\int_0^{\infty} \mathbf{1}\left[|y-1/y| \le R/(n\sqrt{\gamma})\right]dy.\end{aligned}$$

Now fix $\alpha > 0$. Notice that if $0 \le x \le 1$,

$$|x-1/x| \le 2\alpha \implies x \ge \sqrt{\alpha^2+1}-\alpha.$$

And so

$$\begin{aligned}\int_0^1 \mathbf{1}\left[|x-1/x| \le 2\alpha\right] &\le 1+\alpha-\sqrt{\alpha^2+1} \\ &\le C\alpha.\end{aligned}$$

Similarly if $x \ge 1$, then

$$|x-1/x| \le 2\alpha \implies x \le \alpha+\sqrt{\alpha^2+1}.$$

And so

$$\int_1^\infty \mathbf{1}\left[|x - 1/x| \leq 2\alpha\right] \leq \alpha - 1 + \sqrt{\alpha^2 + 1}$$
$$\leq C\alpha.$$

Therefore

$$\sqrt{\gamma}\int_0^\infty \mathbf{1}\left[|x - 1/x| \leq R/(n\sqrt{\gamma})\right] dx \leq C\sqrt{\gamma}\frac{R}{n\sqrt{\gamma}}$$
$$= C/n.$$

□

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