The Shape of the Eigenvectors of the One Dimensional Random Schrödinger Operator

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November 26, 2014

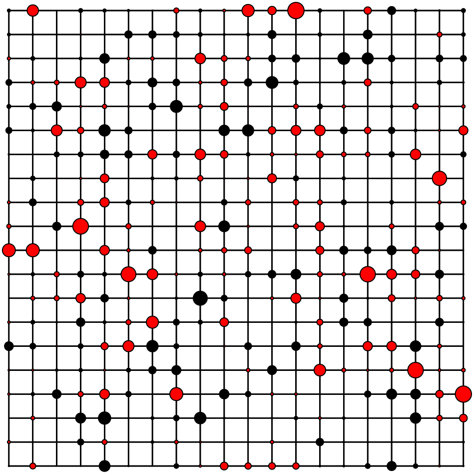
Anderson Model

Anderson (1958) introduced a model of the behaviour of electrons in alloy crystal.

- Crystal: lattice structure of atoms.
- Alloy: random potential at every lattice point.
- Approximations: model interaction of electron with atoms via just an exterior potential, neglect electron-electron interaction.

Electrons move along the lattice subject to the random potential.

Anderson Model on $\mathbb{Z} \times \mathbb{Z}$



Discrete Random Schrödinger operator on $\mathbb{Z}$

$$H = \Delta + \sigma V, \quad \text{on } \ell^2(\mathbb{Z})$$

- Δ is the discrete Laplacian on $\mathbb{Z}$,

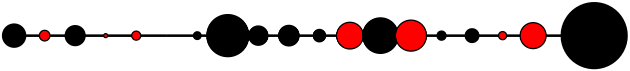
$$(\Delta\phi)_\ell = \phi_{\ell+1} + \phi_{\ell-1}.$$

- V is a random delta potential at each point in $\mathbb{Z}$,

$$(V\phi)_\ell = v_\ell \phi_\ell, \quad ,$$

$$v_\ell \text{ are i.i.d on } \mathbb{Z}, \ \mathbb{E}v_\ell = 0, \ \mathbb{E}v_\ell^2 = 1$$

- $\sigma \in \mathbb{R}$, fixed



Or,

$$H = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & \sigma v_\ell & 1 & & \\ & 1 & \sigma v_{\ell-1} & 1 & \\ & & 1 & \sigma v_{\ell+1} & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

$\sigma = 0$

- no pure point spectrum, no eigenvectors
- spectrum(H) = (-2,2)

Extended eigenvectors:

$$\mu = 2\cos\theta,$$
$$\psi^{\mu}(k) = a\exp(ik\theta) + b\exp(-ik\theta), \;\; k \in \mathbb{Z}.$$

Anderson Localization ($\sigma \neq 0$, random)

Theorem (Goldsheid, Molchanov, Pastur (1977), Kunz, Souillard (1980), Carmona, Klein, Martinelli (1987))

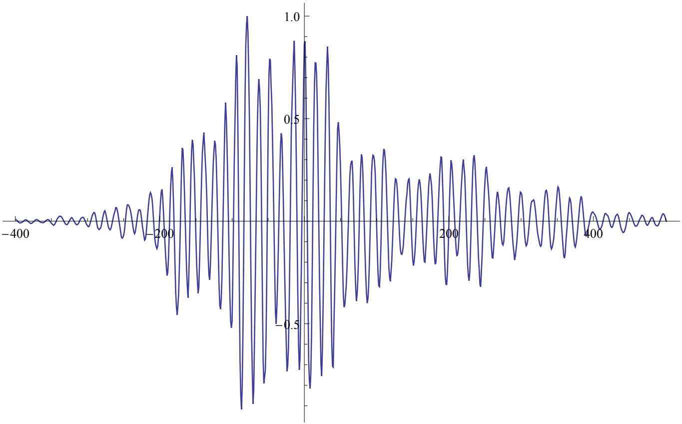
Assume V is random, not concentrated on a single point, and $\mathbb{E}\left[v_0\right]^p < \infty$ for some $p > 0$. Then with probability one,

- *H has pure point spectrum Λ.*
- *H has a complete set of orthonomal eigenvectors $\{\psi^\mu : \mu \in \Lambda\}$ with*

$$|\psi^\mu(k)| \leq C \exp(-m|k - n_0|),$$

(C and n_0 are random)

Anderson Localization



$v_\ell \sim N(0,1)$, $\sigma = 0.1$

Local Eigenvalue Statistics

Restrict H to the finite interval [0,n] with Dirichlet boundary conditions

$$H_n = \begin{pmatrix} \sigma v_1 & 1 & & & \\ 1 & \sigma v_2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & \sigma v_n \end{pmatrix}$$

Empirical Measure

Let Λ_n be the set of eigenvalues of H_n

The (random) empirical eigenvalue measure

$$\mu_n = \frac{1}{n} \sum_{\lambda \in \Lambda_n} \delta_\lambda$$

converges weakly a.s. to the measure $\rho/2\pi$ on $(-2, 2)$ with

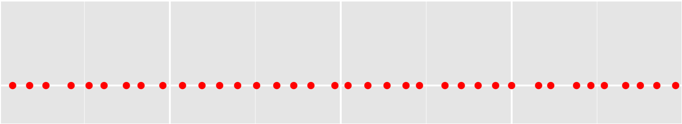
$$\rho(x) = \frac{1}{\sqrt{1 - (x/2)^2}}$$

Local Eigenvalue Point Process

Near $E \in (-2, 2)$, eigenvalue spacing $\sim \rho(E)n$

"zoom in" to see a point process in the limit.

$$n\rho(E)(\Lambda_n - E)$$



Poisson Statistics

Theorem (Minami (1996))

For any $E \in (-2, 2)$, the point process

$$n\rho(E)(\Lambda_n - E)$$

converges in law to a Poisson point process on $\mathbb{R}$ with intensity 1.

Higher Dimensions: $\mathbb{Z}^d$, $d > 1$

- For $\sigma > C_d$
  - Localization (Aizenman and Molchanov, 1993; Fröhlich and Spencer, 1983)
  - Poisson statistics (Minami, 1996)
- For $\sigma < c_d$, conjectures:
  - $d \geq 3$, extended eigenstates and random matrix type statistics of eigenvalues
  - $d = 2$, opinions vary.

Explore Transition between

Anderson localization and local Poisson statistics	$\Longleftrightarrow$	extended states and no local limits

Local Point Process

Take $\sigma = n^{-1/2}$ depending on the size of the interval $[0, n]$.

$$H_n = \begin{pmatrix} v_1/\sqrt{n} & 1 & & & \\ 1 & v_2/\sqrt{n} & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & v_n/\sqrt{n} \end{pmatrix}$$

Local Point Process

Theorem (Kritchevski, Valkó, Virág (2009))

For $0 < |E| < 2$*, the limit of the local eigenvalue point process (properly centred)*

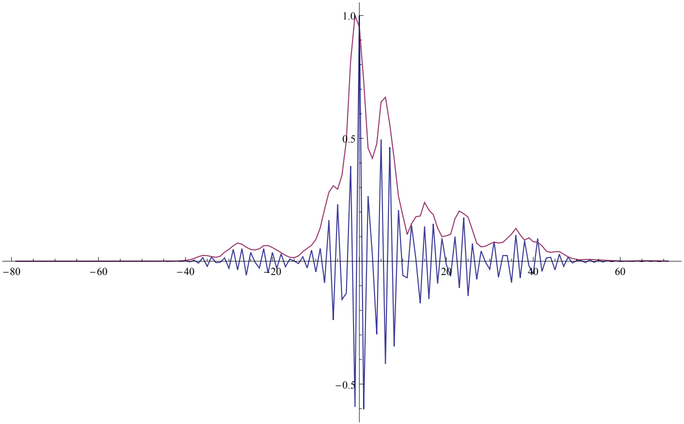
$$n\rho(E)(\Lambda_n - E)$$

converges in law to a point process Sch_E

Eigenvectors: Localized to Extended

Take $\sigma \to 0$.

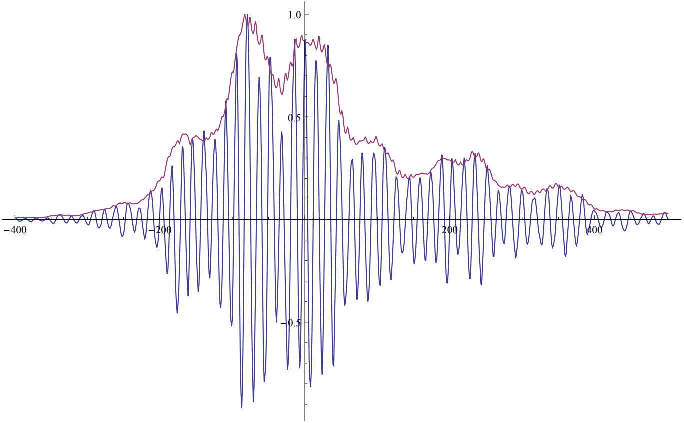
$$H = \Delta + \sigma V,$$
$$v_\ell \sim N(0,1),\ \sigma = 0.5$$



Eigenvectors: Localized to Extended

Take $\sigma \to 0$.

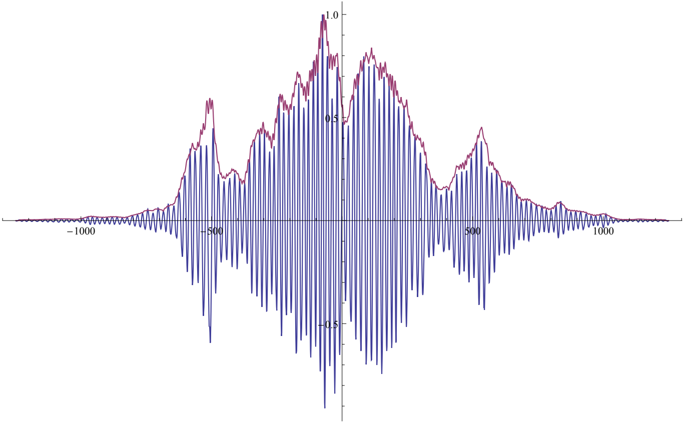
$$H = \Delta + \sigma V, \; \sigma = 0.1$$



Eigenvectors: Localized to Extended

Take $\sigma \to 0$.

$$H = \Delta + \sigma V,\ \sigma = 0.05$$



Eigenvector Shape

$$H = \Delta + \sigma V,$$

- As σ goes to zero the eigenfunctions become more delocalized.
- Localization length $\sim \sigma^{-2}$.
- Goal is to understand the shape of eigenfunctions as the noise goes to zero.

Localization Scale

Back to the finite volume operator,

$$H_n = \begin{pmatrix} v_1/\sqrt{n} & 1 & & & \\ 1 & v_2/\sqrt{n} & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & v_n/\sqrt{n} \end{pmatrix}$$

$\sigma = n^{-1/2}$, localization length $\sim n$.

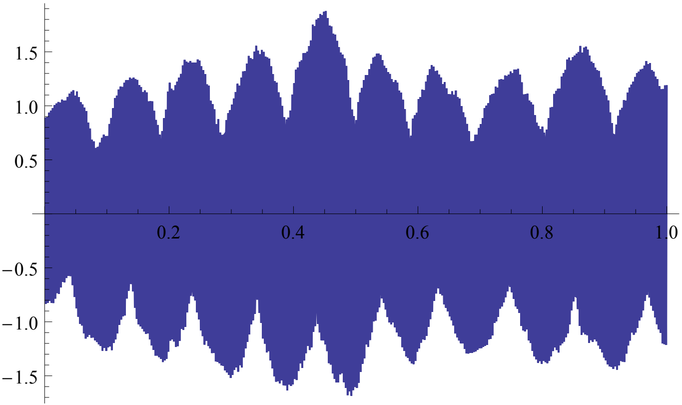
On the scale of localization so we see the main part of the eigenvector.

The Shape of the Eigenvector

Want the scaling limit of eigenvectors as n goes to infinity.

$$\sqrt{n}\psi^{\mu}(\lfloor nt \rfloor), \ \ t \in [0,1]$$

Behaviour too oscillatory.



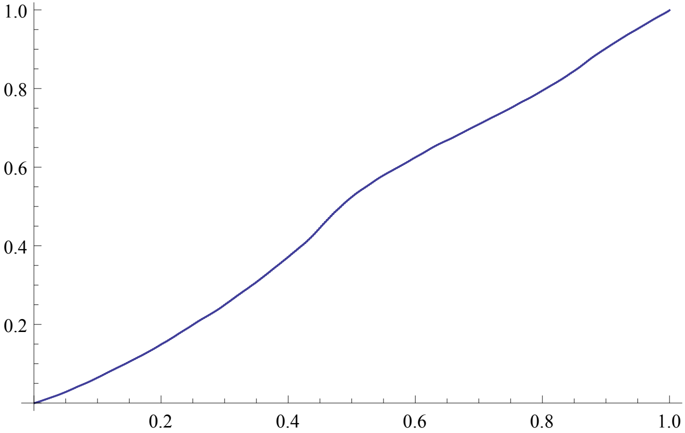
$v_\ell \sim N(0,1), \ n = 1000.$

The Shape of the Eigenvector

Consider instead the measure on $[0,1]$ with density

$$n\,|\psi^{\mu}(\lfloor nt \rfloor)|^2\,dt,\ \ t \in [0,1].$$

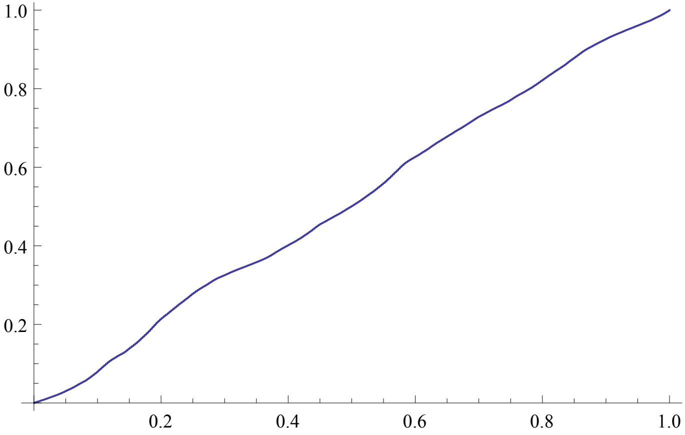
Encodes the distribution of the eigenfunction.



$v_\ell \sim N(0,1),\ n = 1000.$

The Shape of the Eigenvector

$$n\left|\psi^{\mu}(\lfloor nt \rfloor)\right|^2 dt, \;\; t \in [0,1].$$



$$v_\ell \sim N(0,1), \; n = 2000.$$

The Shape of the Eigenvector

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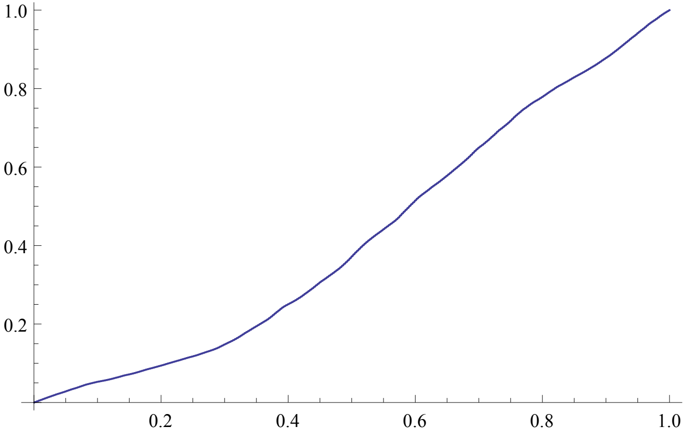

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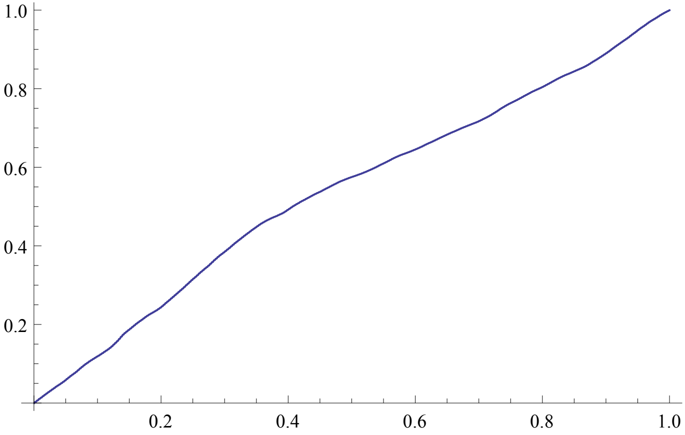
The Shape of the Eigenvector

$$n\,|\psi^{\mu}(\lfloor nt \rfloor)|^2\,dt,\;\; t \in [0,1].$$



$$v_\ell \sim N(0,1),\; n = 2000.$$

Main Theorem

Theorem (Rifkind, Virág)

Pick μ uniformly from the eigenvalues of H_n, then

$$\left(\mu, n\left|\psi^{\mu}\left(\lfloor nt \rfloor\right)\right|^2 dt\right) \Rightarrow \left(E, \frac{S\left(\frac{t-U}{1-(E/2)^2}\right) dt}{\int_0^1 S\left(\frac{s-U}{1-(E/2)^2}\right) ds}\right),$$

where

$$S(t) = \exp\left(-|t|/4 + B_t/\sqrt{2}\right),$$

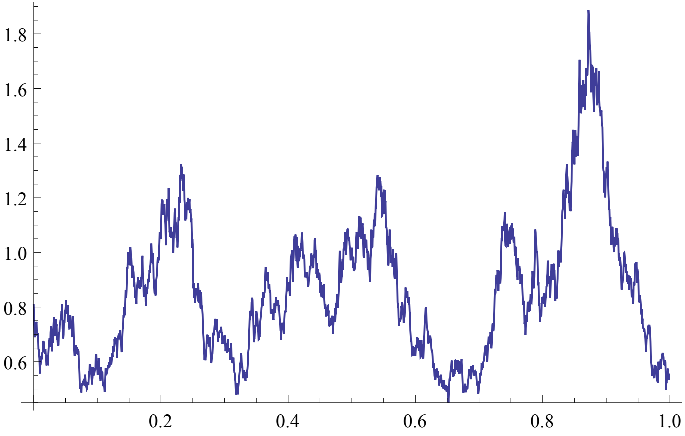
$$E \sim \frac{1}{2\pi\sqrt{1-(E/2)^2}},$$

$$U \sim \mathit{Uniform}[0,1],$$

and B_t, E, U are all independent.

Exponential Brownian Motion

$$\exp\left(-|t-u|/2 + B_{t-u}\right)$$



Exponential Brownian Motion

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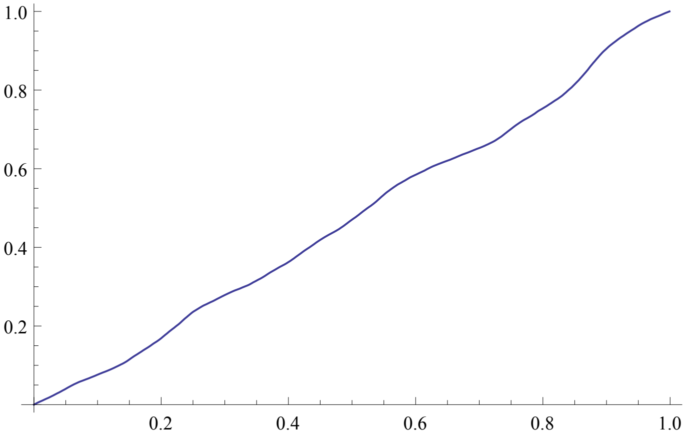

Transfer Matrices

Encode eigenvector, eigenvalue equation in product of 2×2 matrices.
Recall the eigenvalue equation:

$$H_n\phi = \mu\phi,$$
$$\phi_0 = 0 = \phi_{n+1}.$$

$$\implies \phi_{\ell+1} = \left(\mu - \frac{v_\ell}{\sqrt{n}}\right)\phi_\ell - \phi_{\ell-1}$$

We can write this as

$$\begin{pmatrix}\phi_{\ell+1}\\ \phi_\ell\end{pmatrix} = T_\ell^\mu \begin{pmatrix}\phi_\ell\\ \phi_{\ell-1}\end{pmatrix},$$

where

$$T_\ell^\mu = \begin{pmatrix}\mu - \frac{v_\ell}{\sqrt{n}} & -1\\ 1 & 0\end{pmatrix}: \text{ transfer matrix.}$$

Transfer Matrices

Take product of transfer matrices

$$M_\ell^\mu = T_\ell^\mu T_{\ell-1}^\mu \dots T_1^\mu$$

$$\begin{pmatrix} \phi_{\ell+1} \\ \phi_\ell \end{pmatrix} = M_\ell^\mu \begin{pmatrix} \phi_1 \\ \phi_0 \end{pmatrix}.$$

$$M_n^\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \phi_{n+1} \\ \phi_n \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix} \iff \mu \text{ is an eigenvalue,}$$

in which case the corresponding eigenvector is

$$\phi_\ell^\mu = \left(M_{\ell-1}^\mu\right)_{11} \quad \ell = 1, \dots, n.$$

Characterize this random product process to understand the spectral problem for H_n.

Local Transfer Matrix Process

Scale locally around fixed $0 < |E| < 2$.
Zoom in on $\frac{1}{n}$ neighbourhood, $\mu = E + \frac{\lambda}{\rho n}$.
Focus on λ,

$$M_\ell^\lambda = T_\ell^\lambda T_{\ell-1}^\lambda \dots T_1^\lambda,$$

$$T_\ell^\lambda = \begin{pmatrix} E + \frac{\lambda}{\rho n} - \frac{v_\ell}{\sqrt{n}} & -1 \\ 1 & 0 \end{pmatrix}.$$

- $E + \frac{\lambda}{\rho n}$ is an eigenvalue $\iff$ $\left(M_n^\lambda\right)_{11} = 0$
- $\phi_\ell^\mu = \left(M_{\ell-1}^\lambda\right)_{11}$ is the eigenvector

Local Transfer Matrix Process

Want limiting process $\left\{M_\ell^\lambda,\ \ell = 1, \ldots, n\right\}$ as n goes to ∞.

Notice,

$$T_\ell^\lambda = \begin{pmatrix} E + \frac{\lambda}{\rho n} - \frac{v_\ell}{\sqrt{n}} & -1 \\ 1 & 0 \end{pmatrix} \sim T = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}$$

We can diagonalize $T = ZDZ^{-1}$, with

$$Z = \begin{pmatrix} \overline{z} & z \\ 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} \overline{z} & 0 \\ 0 & z \end{pmatrix}, \qquad z = \frac{E}{2} + i\sqrt{1 - E^2/4}.$$

Change of Basis

Look at the M_ℓ^λ process in this basis. For $a, b \in \mathbb{R}$,

$$Z^{-1}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\rho}{2}\begin{pmatrix} ai - bzi \\ \overline{ai - bzi} \end{pmatrix}.$$

$$\implies Z^{-1} M_\ell^\lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} m_\ell^\lambda \\ \overline{m_\ell^\lambda} \end{pmatrix}$$

Eigenvalue hitting equation

- starts at $Z^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\rho}{2}\begin{pmatrix} i \\ -i \end{pmatrix}$
- ends at $Z^{-1}\begin{pmatrix} 0 \\ c \end{pmatrix} = \frac{c\rho}{2}\begin{pmatrix} -zi \\ \overline{-zi} \end{pmatrix}$, $c \in \mathbb{R}$

$\arg m_n^\lambda$ determines if λ is an eigenvalue

Recentering

This is not a well behaved process.

$T_\ell^\lambda \sim T^\ell$ and so in this basis,

$$Z^{-1} M_\ell^\lambda \text{ is a perturbation of } D^\ell$$

Cannot have a limiting process.

$\implies$ recenter: $Q_\ell^\lambda = D^{-\ell} Z^{-1} M_\ell^\lambda$

Local Limiting Diffusion

Theorem (Kritchevski, Válko, Virág)

Fix $0 < |E| < 2$ *and let* $\tau = 1/(1-(E/2)^2)$.

$$(Q^\lambda_{\lfloor nt/\tau \rfloor}, 0 \le t \le \tau) \Rightarrow (Q^{\lambda/\tau}(t), 0 \le t \le \tau),$$

where

$$dQ^\lambda = \frac{1}{2}\left(\begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix} dt + \begin{pmatrix} idB & dW \\ d\overline{W} & -idB \end{pmatrix} \right) Q^\lambda,$$

$$Q^\lambda(0) = Z^{-1}$$

Eigenvectors and Eigenvalues in the Limit

As in the finite case, we can write

$$Q^{\lambda}(t)\begin{pmatrix}1\\0\end{pmatrix} = r^{\lambda}(t)\begin{pmatrix}e^{i\theta^{\lambda}(t)}\\e^{-i\theta^{\lambda}(t)}\end{pmatrix}$$

$\theta^{\lambda/\tau}(\tau)$ determines the limiting point process Sch_E.

$r^{\lambda/\tau}(t)$ determines the limiting eigenvector.

Local Eigenvalue Point Process

Theorem (Kritchevski, Válko, Virág)

Consider the family of SDE's

$$d\theta^{\lambda}(t) = \lambda dt + d\mathcal{B} + Re \left[e^{-i\theta^{\lambda}(t)} d\mathcal{W} \right], \quad \theta^{\lambda}(0) = 0$$

coupled together for all values of $\lambda \in \mathbb{R}$.

$\mathcal{B}$ *and* $\mathcal{W}$ *are standard real and complex Brownian motions.*

For $0 < |E| < 2$ *and with* $\tau = (1/(1-(E/2)^2)$

$$\Lambda_n - \arg(z^{2n+2}) \Longrightarrow Sch_E := \left\{ \lambda : \theta^{\lambda/\tau}(\tau) \in 2\pi\mathbb{Z} \right\}.$$

Local Joint Limit

Theorem

Fix $0 < |E| < 2$, $\tau = 1/(1-(E/2)^2)$, *and* θ *uniform on* $[0, 2\pi]$. *Then,*

$$\left\{ \left(n\rho(E)(\mu - E) + \theta, \frac{n}{\tau} \left| \phi^\mu \left(\left\lfloor \frac{nt}{\tau} \right\rfloor \right) \right|^2 dt \right) : \mu \text{ an eigenvalue of } H_n \right\}$$

converges in law to

$$\mathcal{P}_E = \left\{ \left(\lambda, (r^{\lambda/\tau}(t))^2 dt \right) : \lambda \in Sch_E \right\}.$$

Follows from [KVV] Theorem and more tightness.

Analysis of Limiting Joint Distributions

$$d\theta^\lambda(t) = \lambda dt + d\mathcal{B} + \text{Im}\ \left[e^{-i\theta^\lambda(t)} d\mathcal{W}\right], \quad \theta^\lambda(0) = 0$$

$$d \ln r^\lambda(t) = \frac{dt}{4} + \text{Re}\ \left[e^{-i\theta^\lambda(t)} d\mathcal{W}\right], \quad \ln r^\lambda(0) = 0.$$

For fixed λ, $\ln r^\lambda$ and θ^λ processes are independent.

They are coupled by the fact that

$\{r^{\lambda/\tau}(t),\ 0 \le t \le \tau\}$ is an eigenvector when $\theta^{\lambda/\tau}(\tau) \in 2\pi\mathbb{Z}$

Use co-area formula and Girsanov theory to analyze the distribution of r^λ

Question

How to do this on $\mathbb{Z}$?

$$H = \Delta + \sigma V$$

H has pure point spectrum Λ with ℓ^2 eigenfunctions.

$$\{\psi^\mu : \mu \in \Lambda\}$$

Pick an eigenfunction ψ^λ from the spectral measure at 0,

$$\sum_{\mu \in \Lambda} |\psi^\mu(0)|^2 \, \delta_\mu .$$

Then is it true that

$$\left| \sigma \psi^\lambda \left(t \, \frac{1 - (\lambda/2)^2}{\sigma^2} \right) \right|^2 dt \Rightarrow \exp\left(B_t/\sqrt{2} - t/4 \right) dt \text{ as } \sigma \to 0\,?$$

Thank You